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Social Paradoxes of Majority Rule Voting and Renormalization Group

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Real-space renormalization group ideas are used to study a voting problem in political science. A model to construct self-directed pyramidal structures from bottom up to the top is presented. Using majority rules, it is shown that a minority and even a majority can be systematically self-eliminated from top leadership, provided the hierarchy has a minimal number of levels. In some cases, 70% of the population is found to have zero representation after six hierarchical levels. Results are discussed with respect to the internal operation of political organizations.

KEY WORDS: Renormalization group; majority rule voting; hierarchies; leadership.

Mathematical modeling of social behavior is quite challenging. On one hand, the richness of behavior involved in human situations creates a great deal of complexity. On the other hand, mathematical reductions can be misleading if not dangerous. The frontier between a scientific approach and a political approach is not clear-cut when dealing with social problems. However, such difficulties should not prevent the attempt to study social phenomena on the basis of models as long as there exists no confusion between models and reality.

On these grounds, several attempts have been made to study social behavior using concepts and technique from physics.[1-3] In particular, a model to describe the process of strikes has been presented.[4] More recently, elements for a theory of group decision making have been suggested.[5]

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In this note we study a voting problem in hierarchical structures using some ideas from real-space renormalization groups. Social paradoxes are found to result from the use of majority rule voting.[6]

Having a population distributed among two political directions H and G with respective probabilities p_0 and $(1-p_0)$, we build self-directed hierarchies. Using majority rules to go from one level to the upper one, these structures are found to be always self-directed in the G direction, provided p_0 is smaller than some threshold p_c and also the hierarchy has a minimal number of levels. In particular, it is found that 70% of the population can be self-eliminated from top leadership within six levels and up to 77% with ten hierarchical levels. The case of three political directions is also considered. Results are discussed in the framework of the internal operation of political organizations.

To start, we consider a population distributed among two political directions H and G with respective probabilities p_0 and $(1-p_0)$. People are then randomly selected from the population to form cells, each one containing r persons. To initiate the hierarchy, the bottom level is constituted using the above r-size cells. The first level is obtained then using a voting process within the bottom level. Each r-size cell elects a representative using majority rule and according to the political directions of its own members. Having a probability p_0 to get a H-person in the cell, the probability p_1 to have an H-person elected by a cell is

$$p_1 = R(p_0) \tag{1}$$

where R is a function which accounts for all cell configurations producing an H-majority. For odd sizes there always exists a clear majority. However, when r is even, the half–half case is particular. The usual common sense admits that a half–half vote should have no effect on the actual situation. If the G-direction is already in power, the equality vote is thus taken in favor of a G-person. The function R is given by

$$R(p) = \sum_{l=r}^{m} \frac{r!}{l!\,(r-l)!}\, p^l (1-p)^{r-l} \tag{2}$$

where $m=(r+1)/2$ for odd values of r and $m=(r+2)/2$ for even values of r.

The elected persons now form new r-size cells to constitute the first level. These cells are again built randomly. The same process as before is then used to construct the second level, the probability p_2 to have an H-person elected being $p_2 = R(p_1)$. After n iterations we have an n-level hierarchy with a probability $p_n = R(p_{n-1})$ to have an H-person at top leadership.

At this stage it is of interest to study the variation of p_n as a function of n which results from Eq. (2). The function $R(p)$ is a monotonically increasing function of p. It has two trivial fixed points $p^*=0$ and $p^*=1$. In between there exists a nontrivial fixed point which is $p_r^*=1/2$ for odd r. When r is even we have $1/2<p_r^*<0.77$ (see Table I) with $p_4^*=(1+\sqrt{13})/6$ and $p_r^*\to 1/2$, $r\to +\infty$. Analysis of these fixed points shows that p_r^* is unstable, while $p^*=0$ and $p^*=1$ are both stable (see Figs. 1 and 2).

Therefore, starting from $p_0<p_r^*$, the voting process generates a flow toward the stable fixed point $p^*=0$. This means in particular the elimination of the H-direction from the top leadership. However, this effect must occur for a small number of iterations to be relevant. In dealing with critical phenomena we are interested in the vicinity of the unstable fixed point. Here we are interested in the stable fixed point. Moreover, in physics, iteration of Eq. (2) is a mathematical trick to account for long-range correlated fluctuations in the vicinity of a phase transition. The number of iterations is of no significance. Here it is rather different. Each iteration of Eq. (2) means an additional hierarchical level, i.e., more people in the structure. Giving a p_0, the question then arises of calculating the threshold value n of hierarchical levels at which the probability p_n to get an H-person elected is practically zero. Giving some ε of order 10^{-4}, we want to find n such that $p_m \leqslant \varepsilon$ for $m \geqslant n$. A hierarchy with n levels will thus always appear to be self-directed at its top in the G-direction.

We now present exact numerical results. Plugging some p_0 in Eq. (2), we get p_1 and then iterate the process up to values of order 10^{-4} for the probabilities. In Table II we report variations of p under repeated voting starting from $p_0=0.4995$ for sizes $r=3$, 5, 7, 9, 11, 13, and 15. From the results it is seen that only a few iterations are required to self-eliminate the H-direction, which makes the model realistic. Moreover, for a fixed initial p_0 the number of levels to get practically zero decreases with increasing r. For instance, an H-direction supported by 44% of the population is self-eliminated within four hierarchical levels using groups of seven persons. Increasing the size cell to 15 persons reduces the number of levels to three (see Table II).

Table I. Unstable Fixed-Point Values as Function of Cell Size

r	4	6	8	10	12	14	16	18	20
p_r^*	0.77	0.65	0.60	0.58	0.56	0.55	0.54	0.54	0.53

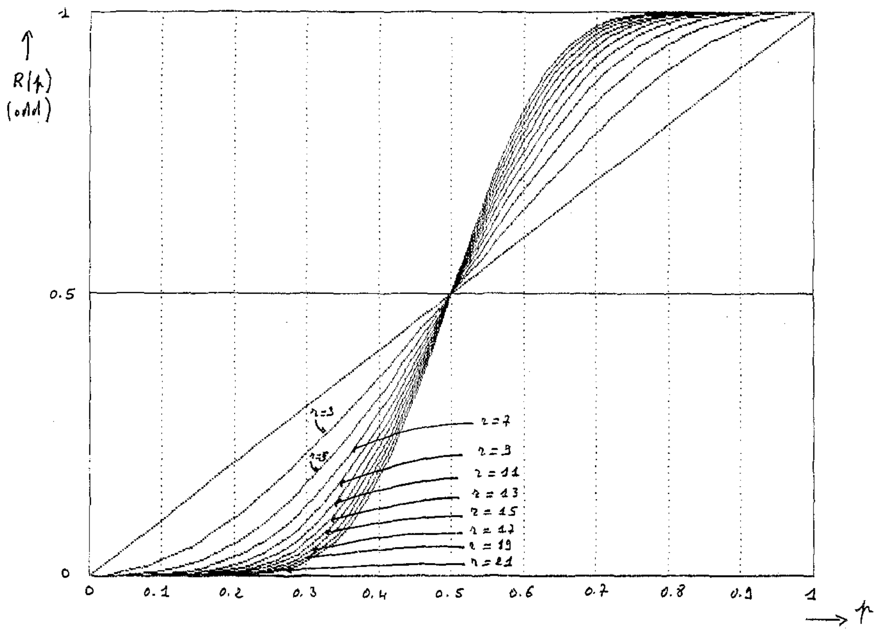

Fig. 1. Functions $y = R(p)$ for various odd values of r.

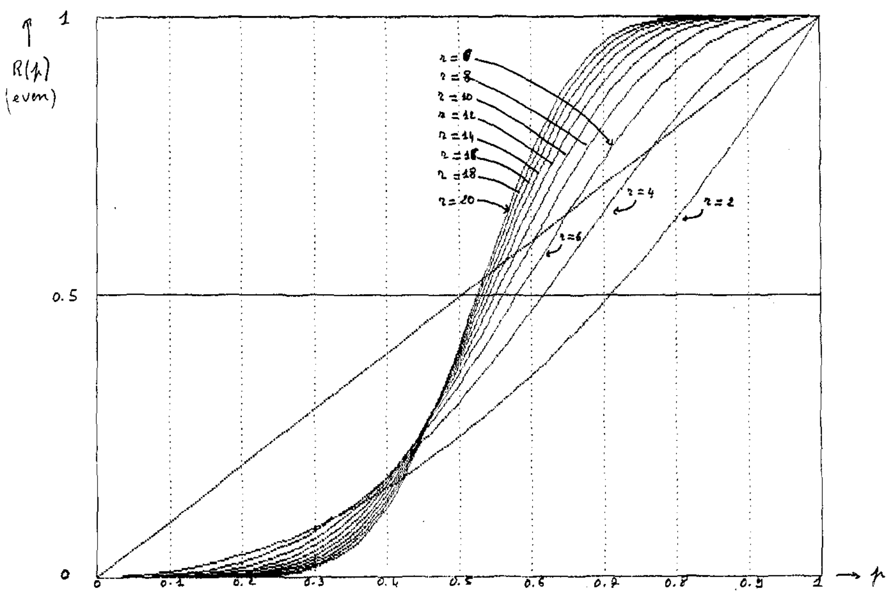

Fig. 2. Functions $y = R(p)$ for various even values of r.

Table II. Iteration of Eq. (2) with Initial Value $p = 0.4995$ for Various Values of r

n	$r=3$	$r=5$	$r=7$	$r=9$	$r=11$	$r=13$	$r=15$
0	0.4995	0.4995	0.4995	0.4995	0.4995	0.4995	0.4995
1	0.4992	0.4990	0.4989	0.4987	0.4986	0.4985	0.4984
2	0.4988	0.4982	0.4976	0.4969	0.4963	0.4957	0.4950
3	0.4983	0.4967	0.4947	0.4925	0.4900	0.4873	0.4844
4	0.4974	0.4938	0.4885	0.4816	0.4731	0.4630	0.4513
5	0.4962	0.4884	0.4749	0.4549	0.4277	0.3928	0.3505
6	0.4943	0.4782	0.4453	0.3903	0.3109	0.2127	0.1141
7	0.4914	0.4593	0.3819	0.2467	0.0912	0.0099	0.0000
8	0.4871	0.4240	0.2556	0.0464	0.0001	0.0000	
9	0.4807	0.3598	0.0759	0.0000	0.0000		
10	0.4711	0.2506	0.0009				
11	0.4568	0.1041	0.0000				
12	0.4354	0.0096					
13	0.4036	0.0000					
14	0.3572						
15	0.2917						
16	0.2056						
17	0.1095						
18	0.0333						
19	0.0032						
20	0.0000						

In the case of even sizes the above effects are much more drastic. The number of levels is smaller than for odd sizes of cells. Moreover, a majority can now be self-eliminated. Results are presented in Table III. Using four-size cells, 49.95% are self-eliminated within three levels, while six levels are enough to neutralize 71% of H-support in the population. However, in the present even case, the number of levels needed to neutralize a given H-support is an increasing function of the cell size, in contrast to the above odd case (see Table III).

To formulate the above results analytically, we need a formula giving n as a function of r, p_0, and $p_n = \varepsilon$. We first perform a Taylor expansion of $p_n = R(p_{n-1})$ around $p^* = 0$, which is indeed Eq. (2). To lowest order in p_{n-1} we have

$$p_n \simeq \mu p_{n-1}^m \tag{3}$$

where $\mu = r!/[m!\,(r-m)!]$. Iterating then Eq. (3) n times gives

$$\mu^{1/(m-1)} p_n \simeq (\mu^{1/(m-1)} p_0)^{(m^n)} \tag{4}$$

Table III. Iteration of Eq. (2) with Several Values for $r=4$, 6, 8, and 10

n	$r=4$	$r=6$	$r=8$	$r=10$
0	0.7600	0.6500	0.4995	0.4995
1	0.7550	0.6471	0.3622	0.3757
2	0.7467	0.6412	0.1207	0.1285
3	0.7328	0.6292	0.0010	0.0006
4	0.7089	0.6047	0.0000	0.0000
5	0.6674	0.5542		
6	0.5940	0.4499		
7	0.4648	0.2551		
8	0.2616	0.0404		
9	0.0576	0.0000		
10	0.0007			
11	0.0000			

from which we get

$$n=\frac{1}{\ln(m)}\ln\left[\frac{\ln(\mu^{1/(m-1)}\varepsilon)}{\ln(\mu^{1/(m-1)}p_0)}\right] \tag{5}$$

However, from Eq. (4) it is seen that Eq. (5) holds only in the range $\mu^{1/(m-1)}p_0<1$. Otherwise p_n would be larger than p_0, which is not consistent with the existence of one unstable fixed point p_r^* when $p_0<p_r^*$. For larger values of p_0 the expansion must be performed around p_r^*, which gives at lowest order

$$p_k \simeq p_r^* - (p_r^* - p_{k-1})\lambda \tag{6}$$

where $\lambda=(dR/dp)/p_r^*$.

At this stage we are approximating Eq. (2) by a straight line [Eq. (6)] and by a power law [Eq. (3)] in the vicinity of, respectively, p_r^* and zero. To optimize the range of validity of these approximations in the whole range $0<p<p_r^*$, we calculate the value $\bar{p}$ which minimizes the distance between Eqs. (3) and (6) to get

$$\bar{p}=\left(\frac{\lambda}{m\mu}\right)^{1/(m-1)} \tag{7}$$

Starting with a p_0 larger that $\bar{p}$ to go down to p_n, we have first to iterate Eq. (6) up to $\bar{p}$ to get a number of iterations

$$k=-\frac{\ln(p_r^*-p_0)}{\ln\lambda}+\frac{\ln(p_r^*-\bar{p})}{\ln\lambda} \tag{8}$$

The second step is to use Eq. (5) to go from $\bar{p}$ down to $p_n = \varepsilon$. We thus obtain for the total number of iterations

$$n = -\frac{\ln(p_r^* - p_0)}{\ln \lambda} + n_0 \tag{9}$$

where n_0 is a number given by

$$n_0 = \frac{\ln(p_r^* - \bar{p})}{\ln \lambda} + \frac{1}{\ln(m)} \ln \left[\frac{\ln(\mu^{1/(m-1)}\varepsilon)}{\ln(\mu^{1/(m-1)}\bar{p})} \right] \tag{10}$$

Rounding to an integer, Eq. (9) gives the number of hierarchical levels needed to self-eliminate an H-direction with some $p_0 > \bar{p}$ support in the population. When $p_0 < \bar{p}$ the number of levels is given by Eq. (5). It is worth noticing, however, that putting $n_0 = 0$ in Eq. (9) makes it a good approximation for $p_0 < \bar{p}$, if we take $n+1$ instead of n.[6]

To illustrate Eqs. (5) and (9), we calculate the various quantities involved for both cases $r=3$ and $r=4$. When $r=3$, we get $m=2$, $p_3^* = 1/2$, $\mu = 3$, $\lambda = 3/2$, $\bar{p} = 0.25$, and $n_0 = 1.398$. The actual value of n_0 depends very slightly on the chosen value of ε. Here we take $\varepsilon = 0.0001$. From Eq. (9) we thus find $n = 11.17$ for $p_0 = 0.481$ and $n = 6.19$ for $p_0 = 0.357$, where rounded integers are just the exact values 11 and 6 (see Table II). For $p_0 = 0.206$, $n = 4.07$ from Eq. (5), which gives the exact result 4. Using Eq. (9) with $n_0 = 0$ gives $n = 3.02$, which corresponds to 4 by rounding $n+1$.

In the case $r=4$, we have $m=3$, $p_4^* = 0.77$, $\mu = 4$, $\lambda = 1.64$, $\bar{p} = 0.37$, and $n_0 = 1.19$. From Eq. (9) we find $n = 6.90$ for $p_0 = 0.709$, and Eq. (5) gives $n = 3.07$ for $p_0 = 0.262$. Both associated integers 7 and 3 are the exact results (see Table III). For the last case, uisng Eq. (9) with $n_0 = 0$ gives $n = 1.36$. Rounding $n+1$ to an integer part gives $n = 2$.

Most social organizations, in particular political ones, have a fixed structure with a given number of hierarchical levels. Therefore, to discuss social implications of the model, we have to invert Eq. (9) to obtain p_0 as function of n, which is

$$p_0 = p_r^* - \lambda^{n_0 - n} \tag{11}$$

when $n > n_0$ (satisfied for most cases). Otherwise Eq. (5) must be used. Equation (11) is the instrumental result of our model. It defines the threshold p_0 of oppositional support in an n-level party which does not put at stake the actual leadership.

From Eq. (11) it is seen why a strong increase to H-support from p' to p does not affect the party as long as $p < p_0$. In parallel, a small increase which makes $p > p_0$ has drastic effect with a complete change of top leadership in favor of the H-direction at once. Two political practices can also be understood on the basis of Eq. (11). First, the use of individual exclusion appears to be an effective way to keep $p < p_0$ in cases where there exists a trend toward $p > p_0$. Second, it is found that imposed quotas to help some groups in the organization, such as women or ethnic minorities, have no effect in generating their representation at the top leadership, since these quotas are always smaller than p_0.

At this stage the natural step is to extend the model to more than two political directions which corresponds indeed to real situations. The main difference is the increase in dimensions of the flow diagrams. For two directions the flow diagram is one-dimensional (see Figs. 1 and 2). In the case of N directions the associated flow diagram becomes $(N-1)$ dimensional. To illustrate the process, we mention the simple case of three directions, I, H, and G, with 15-size cells. We assume a political context where I-persons and H-persons vote only for themselves or, if they have no majority, for the G-direction. It corresponds to frequent political situations with two major opposite parties and a small one in between. In such a context p^H_{j+1} and p^I_{j+1} are then given by Eq. (2) with $r = 15$. From the analysis of the $r = 15$, two-directional case it follows that the threshold for both H and I directions is $p^*_{15} = 1/2$. This means in particular that the H and I directions will be self-eliminated as long as they do not have an initial support of more than 50%. The striking consequence is that the G-direction will get complete control of the top leadership even with 2 or 3% of support in the population. To illustrate this case, we consider an initial population with $p^H_0 = 0.48$, $p^I_0 = 0.45$, and $p^G_0 = 0.07$. Within five levels (see Table II), $p^H = p^I = 0$, while $p^G = 1$. This illustration shows why small parties which compromise with two major parties which are opposed can get overweighted representation at the top leadership. A more detailed study of multidirectional situations is underway.

In this note we have shown how some technique and ideas from statistical physical can be used to study a voting problem. Self-oriented pyramidal structures have been obtained with hierarchical levels leading to the top leadership. The topology of these hierarchies exhibits some similarity with so-called Cayley trees or Bethe lattices. However, they are also different, since the use of majority rules within cells is equivalent to interactions among sites of the same generation, which are indeed forbidden in Cayley trees. Although the model is a rather crude approximation of social reality, it emphasizes some basic features associated with majority rule voting. Nonintuitive results were found, in particular, the self-elimination of a majority within a small number of hierarchical steps. The model should also apply to other fields, such as biology and economics.

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REFERENCES

1. W. Weidlich, *Br. J. Math. Stat. Psychol.* **24**:251 (1971).
2. E. Callen and D. Shapero, *Phys. Today* **23** (July 1974).
3. J. Gersten and R. W. Bologh, Preprint.
4. S. Galam, Y. Gefen, and Y. Shapir, *J. Math. Sociol.* **9**:1 (1982).
5. S. Galam and S. Moscovici, *Eur. J. Soc. Psychol.*, in press.
6. S. Galam, *J. Math. Psychol.* **30**:426 (1986).

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